Dynamics of Spatial Soliton in a Gradient Refractive Index Waveguide with Nonlocal Nonlinearity

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ABSTRACT

We study the propagation of a spatial soliton in a triangular gradient refractive index (GRIN) waveguide with nonlocal nonlinearity. Dynamics of such soliton propagation are predicted analytically using equivalent-particle approach. It is shown that without nonlocal nonlinearity the soliton oscillates periodically and symmetrically with the oscillation center exactly at the center of waveguide. Weak nonlocal nonlinearity leads to an asymmetric oscillation of the soliton and also shifts the oscillation center. Stronger nonlocal nonlinearity will produce a soliton exit from the waveguide where the soliton remains stable. Furthermore, a soliton with higher amplitude causes a much bigger nonlocal effect. The dynamics of the soliton are also simulated numerically. The results of our simulations agree very well with our analytical prediction.

Keywords: Spatial soliton, GRIN waveguide, Nonlocal nonlinearity, (Modified-) Nonlinear Schrödinger equation, Equivalent-particle approach

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1. INTRODUCTION

Spatial solitons are self-trapped optical beams that propagate without change through a nonlinear medium due to the diffraction associated with the finite size of the wave is balanced by the self-focusing induced by nonlinearity; see e.g. [Boardman and Sukhorukov, 2000]. Spatial solitons have been extensively studied in many areas of physics. The interest to propagation of such solitons is driven mainly by their rich potential for all optical switching, and other manipulating light by light itself; see e.g. [Scheuer and Orenstein, 1999; Wu, 2004; Yabu et al., 2002; Wu, 2005].
In a local Kerr nonlinear medium, it is well known that spatial soliton is alike a particle moving with constant velocity. The transverse velocity of spatial soliton is disturbed when it propagates in an inhomogeneous refractive index medium, i.e., a medium with a transverse gradient refractive index (GRIN) distribution. In the case of weak GRIN, the soliton can still be treated as a particle. In this way [Aceves et al., 1989] developed an equivalent particle approach to study the dynamics of spatial solitons due to the interface separating nonlinear media. This method has been applied to study the soliton properties in a local Kerr nonlinear medium with symmetrical linear refractive index variation, see e.g. [Garzia et al., 1997; Suryanto and van Groesen, 2001; Suryanto and van Groesen, 2006]. In those symmetrical GRIN waveguide, a spatial soliton exhibits an oscillatory behavior where its oscillation period is determined by the soliton amplitude as well as by the waveguide parameters.

Most of the above mentioned studies are directed to soliton in a local nonlinear medium. But, under appropriate conditions, the nonlinear response of media might be significantly nonlocal in the sense that it importantly affects the properties of soliton [Krolikowski and Bang, 2000]. For example, nonlocality may induce phenomenon of spatial soliton self-bending [Petter et al., 1999; Izdebskaya et al., 2010]. [Kartashov et al., 2004] have investigated the combine effects of a transverse periodically modulated index and the nonlocality. The soliton in such medium remains located (trapped) in the guiding channel of the harmonic lattice or travels along the lattice (i.e., soliton self-bend). Recently [Dong and Wang, 2006] studied the propagation of soliton within one period of such lattice. In that study, they consider a medium which has a transverse symmetrical parabolic refractive index distribution. It was shown that for weak nonlocality the soliton remains stable and may oscillate periodically in the waveguide. Stronger nonlocality leads to instability of the oscillatory soliton, i.e. both periodical oscillation and soliton stability are destroyed. Here we will study the dynamics of soliton in a GRIN waveguide with nonlocal nonlinear response where the linear refractive index has a triangular profile. It will be shown that when the nonlocality is weak, the soliton in a triangular GRIN waveguide behaves similarly with that in a parabolic GRIN waveguide. However, for relatively large nonlocality, the behavior of soliton in a triangular and parabolic GRIN waveguide may completely be different.

2. GOVERNING EQUATION

The governing equation which describes the beam propagation in nonlinear waveguide can be derived from the Maxwell equation. Under the paraxial simplification, the propagation of optical waves along the z axis in a slab waveguide with homogeneous linear refractive index and local Kerr nonlinearity is described by the nonlinear Schrödinger equation (NLSE):

\[
i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0 \tag{1}
\]

where \( u \) is the dimensionless slowly varying amplitude envelope, the longitudinal \( z \) and transverse \( x \) coordinates are scaled to the diffraction length and input beam width, respectively.
If we consider the effects of inhomogeneous linear refractive index in a transverse direction and first-order nonlocal contribution to nonlinear response, the beam propagation equation is described by the modified nonlinear Schrödinger equation (m-NLSE), see [Kartashov et al., 2004; Dong et al., 2006]:

\[
i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + |u|^2 u = Vu. \tag{2}
\]

\(V\) in equation (2) can be written as \(V = V_1 + V_2\) where \(V_1 = -\Delta n(x)\) and \(V_2 = \mu \frac{\partial |u|^2}{\partial x}\) are the contributions of linear refractive index inhomogeneity and nonlocality, respectively. Here, the parameter \(\mu\) describes the strength of the nonlocal component of nonlinear response. In order to be consistent with the fact that in practice the nonlocal contribution is small compared to the local one, we assume that the parameter \(\mu\) is small. The function \(\Delta n(x)\) in equation (2) stands for profile of GRIN distribution. In this paper we consider a case where the GRIN distribution has a triangular profile is

\[
\Delta n(x) = \begin{cases} 
0, & x < -b \\
\Delta n_0 \left( 1 + \frac{x}{b} \right), & -b \leq x < 0 \\
\Delta n_0 \left( 1 - \frac{x}{b} \right), & 0 \leq x < b \\
0, & x \geq b 
\end{cases} \tag{3}
\]

where \(2b\) is the total width of the waveguide and \(\Delta n_0\) is the maximum index variation. Notice that the waveguide center is located at \(x = 0\).

3. EQUIVALENT PARTICLE THEORY

To analyze the dynamic of a spatial soliton in the GRIN waveguide with nonlocal nonlinearity, a soliton is first considered as a particle whose position is given by \(\vec{x}(z)\) where \(z\) is treated as the time variable. To find an equation for the motion of the particle (beam) we then define the center of the beam as \(\vec{x}(z) = p^{-1} \int_{-\infty}^{\infty} |u|^2 dx\), where \(p = \int_{-\infty}^{\infty} |u|^2 dx\) is the beam energy (power) that conserves during the propagation. From equation (2), we obtain that the velocity and the acceleration of the beam are respectively given by

\[
v(z) = \frac{d\vec{x}(z)}{dz} = \frac{1}{2} i \int_{-\infty}^{\infty} \frac{\partial u^*}{\partial x} u - u \frac{\partial u^*}{\partial x} dx \tag{4}
\]

\[
\frac{d^2\vec{x}(z)}{dz^2} = \frac{dv(z)}{dz} = -\frac{1}{p} \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} |u|^2 dx. \tag{5}
\]
By assuming that the intensity $|d|^2$ moves as a single unit and is a function of $x - \tau(z)$, equation (5) can be written as [Aceves et al., 1989]:

$$a(\tau) = \frac{d^2 \tau(z)}{dz^2} = -\frac{\partial U(\tau)}{\partial \tau},$$

(6)

where $-U$ is the integral of the right hand side of equation (5) with respect to $\tau$. Equation (6) describes the movement of a particle in a Newton’s potential $U$.

As mentioned previously, we assume that the refractive index perturbation due to GRIN and nonlocal response is small. We therefore apply a quasi-homogeneous approximation and the perturbed soliton solution becomes [Suryanto and van Groesen, 2001]:

$$u(x, z) = q \text{sech}(q(x - \tau)) \exp(i(\nu(z)x + \sigma(z))),$$

(7)

We notice that the velocity is now a function of propagation distance $z$, i.e. $\nu(z) = d\tau(z)/dz$ and $d\sigma(z)/dz = \left(q^2 - \nu(z)^2\right)/2$. Upon substituting the soliton trial function (7) into equation (5) or (6), one gets the explicit formula of Newton’s acceleration

$$a(\tau) = a_1(\tau) + a_2(\tau),$$

(8)

where $a_1(\tau) = \frac{\Delta n_0}{b} \frac{\exp(2q\tau)\exp(2bq) - 1}{\exp(2q\tau + 1)\exp(2q(\tau + b) + 1)\exp(2bq) + \exp(2q\tau)}$ and $a_2(\tau) = \frac{8\mu q^4}{15}$ are respectively the Newton’s acceleration due to linear refractive index variation and nonlocality. From equation (6) and (8) we also obtain the Newton’s potential

$$U(\tau) = U_1(\tau) + U_2(\tau),$$

(9)

where $U_1(\tau) = -\frac{\Delta n_0}{2bq} \ln \left( \frac{(\exp(2bq) + \exp(2q\tau])(\exp(2q(\tau + b)) + 1)}{(\exp(2q\tau + 1))} \right)$ and $U_2(\tau) = \frac{8\mu q^4\tau}{15}$ correspond to $a_1(\tau)$ and $a_2(\tau)$, respectively. An important remark is that both $a_1(\tau)$ and $U_1(\tau)$ do not depend on nonlocality and similarly both $a_2(\tau)$ and $U_2(\tau)$ do not depend on linear index variation. This fact can be understood from the linearity of derivative and integral operators in (5). This result is in contrast with the formulae given by [Dong and Wang, 2006] where the contribution of nonlocality on both acceleration and potential in [Dong and Wang, 2006] are also affected by the width of waveguide ($b$).

One can also observed in equation (8) and (9) that the transversal acceleration and potential depend linearly on the slope of the GRIN ($\Delta n_0/b$) as well as on the strength of nonlocality ($\mu$). On the other hand, the contribution of soliton amplitude $q$ in both Newton’s acceleration and potential is proportional to $q^4$ and therefore if the soliton amplitude is large then the contribution of nonlocality will dominate the soliton dynamics.
In Figure 1 we show the acceleration and potential profiles of a soliton with amplitude \( q = 1 \) placed in the waveguide with nonlocal nonlinearity where \( \Delta n_0 = 0.1 \) and \( b = 5 \) for different values of \( \mu \). In the absence of nonlocality, the acceleration of equivalent particle is an antisymmetric function of \( x \); positive on the left side of \( y \)-axis and negative on the other side. Hence, if a soliton beam is launched at a position shifting from the center of waveguide with a zero velocity, it will subject to an effective force and oscillates periodically inside the waveguide. A positive nonlocality shifts up the acceleration and destroys the symmetricity of the potential profile. Indeed, a relatively small nonlocal nonlinearity will
disturb the symmetricity of the soliton oscillation. Stronger nonlocal nonlinearity will always produce a positive acceleration and therefore produces an exiting soliton from the waveguide.

We mentioned previously that the contribution of soliton amplitude \( q \) in the nonlocality effects is proportional to \( q^4 \). This contribution shifts up the profile of acceleration. The acceleration profile is much more shifted up if the input soliton has higher amplitude; see Figure 2(a). The shifting up of the acceleration profile corresponds to the disturbance of the Newton potential well. Figure 2(b) shows that for a relatively high amplitude soliton, the potential well is destroyed such that it does not support the oscillation behavior. For the latter case, the soliton will be forced to exit from the waveguide.

![Figure 3](image)

**Figure 3.** Propagation of a soliton with amplitude \( q = 1 \) and initial position \( x_0 = -2.5 \) in a waveguide with \( \Delta n_0 = 0.1, b = 5 \) and different \( \mu \). In (a) \( \mu = 0.0 \), (b) \( \mu = 0.01 \), (c) \( \mu = 0.025 \), and (d) \( \mu = 0.05 \).

4. SIMULATION OF THE DYNAMICS OF A SPATIAL SOLITON

In order to confirm our analytical prediction in the previous section, we perform some numerical simulations of a spatial soliton propagating in GRIN waveguide with nonlocal nonlinearity using implicit Crank-Nicolson scheme [Darti et al., 2011]. The initial conditions for these simulations are of the form \( (7) \) with zero velocity and initial position at \( x = x_0 \). The maximum linear refractive index variation (\( \Delta n_0 \)) is taken to be 0.1. In Figure 3 we show the evolution of soliton in a waveguide with \( b = 5 \) for several values of \( \mu \). In the absent of nonlocality (\( \mu = 0 \)), during the propagation, the soliton oscillates inside the waveguide symmetrically; see Figure 3(a). If we take \( \mu = 0.01 \) the soliton still oscillates in the waveguide but it is not symmetric around the oscillation center anymore. Furthermore, the center of oscillation is slightly deviated from the center of waveguide; see Figure 3(b). For larger values of \( \mu \), the soliton oscillation may completely be destroyed and produce an exiting soliton from the waveguide. This phenomenon is depicted in Figure 3(c-d). All these numerical observations are in accordance with the analysis presented in previous section. Furthermore, it is also noticed that in both
cases (oscillating and exiting soliton), the soliton remains stable. This phenomenon is different with the soliton in a parabolic GRIN waveguide with nonlocal nonlinearity. In particular, for the case of stronger nonlocality, soliton in a parabolic GRIN waveguide oscillates unperiodically in the waveguide and the soliton shape is not stable; see [Dong and Wang, 2006].

![Figure 4](image)

**Figure 4.** Propagation of a soliton with initial position $x_0 = -2.5$ in a waveguide with $\Delta n_0 = 0.1$, $b = 10$ and $\mu = 0.01$ for (a) $q = 0.5$, (b) $q = 1.0$, and (c) $q = 1.5$.

To understand the role of soliton amplitude, we perform some simulations of soliton propagation in waveguide with $\Delta n_0 = 0.1$, $b = 10$ and $\mu = 0.01$ by varying $q$. The influence of soliton amplitude on the dynamics of soliton is clearly seen in Figure 4. A relatively small amplitude soliton oscillates in the waveguide since the refractive index perturbation caused by GRIN is stronger than that by nonlocality. Nevertheless, the oscillation is disturbed by nonlocality which can be seen from the fact that the oscillation center deviates from the center of waveguide and the velocity is not symmetric about the center of oscillation. Such phenomenon is more pronounced for soliton with larger amplitude. If the soliton amplitude is further increased the soliton does not oscillate and exit from the waveguide (but the soliton remains stable). This shows that the soliton with larger amplitude leads to a much bigger nonlocal effect. This phenomenon perfectly agrees with the prediction of equivalent-particle approach.

5. CONCLUSION

We have investigated the dynamics of spatial soliton in a waveguide with nonlocal nonlinearity where the linear refractive index has a triangular profile. Our analytical and numerical studies show that when the nonlocality is relatively small then the soliton oscillates asymmetrically where the center of oscillation deviates from the waveguide center. Moreover, the higher amplitude soliton is more sensitive to nonlocality than the lower one and therefore it can destroy the potential well of the particle-like soliton. In this case, the soliton exits from the waveguide and it remains stable.
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7. REFERENCES


