A nonstandard finite difference (NSFD) scheme for a harvesting Leslie-Gower equations is constructed. It is shown that the obtained difference system has the same dynamics as the original continuous system, such as positivity of solutions, equilibria and their local stability properties, irrespective of the size of numerical time-step. To illustrate the analytical results, we present some numerical simulations.

**Keywords:** predator-prey model; difference equation; unconditionally positive; dynamically consistent; asymptotically stable, discrete Leslie-Gower equation

**AMS Subject Classification:** 03H05; 37N25; 37N40; 92D40

1. **Introduction**

The interaction between predator and its prey is one of dominant subjects in both ecology and mathematical ecology. One of typical predator-prey model is the Leslie-Gower equations. The Leslie-Gower equations assume that the reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. This assumption can be seen from the carrying capacity of the predator environment which is proportional to the number of prey. This shows that there are upper limits to the rates of increase of both prey and predator, which are not recognized in the Lotka-Volterra model [14]. From point of view of human needs, the exploitation of biological resources and the harvest of population are commonly practiced in fishery, forestry and wildlife management [10]. In this respect, Zhang et al. [21] considered that the predator and prey species are both of commercial importance and therefore they are subjected to constant effort harvesting. Based on the Leslie-Gower model and this assumption, they formulated the following continuous model

$$\frac{dH}{dt} = (r_1 - a_1 P - b_1 H) H - c_1 H, \quad (1a)$$
where $H$ and $P$ are respectively the density of prey species and the predator species at time $t$. Here, all the parameters are positive. Those parameters are defined as follows: $r_1$ and $r_2$ are the intrinsic growth rate of prey and predator; $b_1$ measures the strength of competition among individuals of prey; $a_1$ is the predation rate; $a_2$ is a measure of the food quantity that the prey provides converted to predator birth; $P/H$ is the Leslie-Gower term which measures the loss in the predator population due to rarity of its favorite food; $c_1$ and $c_2$ are the constant effort harvesting of prey and predator, respectively. To ensure the sustainable development of both species, it is assumed that $0 < c_i < r_i$, $i = 1, 2$.

For practical purposes, it is often necessary to obtain solutions of system (1) which describe the evolution of both prey and predator for different times. Since the exact analytical solution of system (1) is generally difficult or even impossible to be determined, a numerical discretization of the model is needed. The obtained discrete model should preserve the dynamical properties of the original continuous model as much as possible. It is known that many standard methods (including forward Euler method, Runge-Kutta method, etc.) may not satisfy this requirement. Those methods may lead to incorrect dynamical properties such as having negative solutions, converging to wrong equilibrium points; cause chaotic dynamics or even numerical instabilities [8, 9, 17–20]. To avoid such dynamic inconsistency, in this paper we implement a Nonstandard Finite Difference (NSFD) method to discretize system (1). This technique is originally developed by Mickens [11–13] and has been applied to various biological systems [2–6, 15–17, 19, 20] in which the resulting discrete models are shown to be very effective in preserving the dynamical properties of the original continuous models.

2. Dynamical properties of continuous model

Before constructing the numerical scheme, we discuss the properties of system (1). Due to biological nature and to avoid the singularity, the solutions of system (1) has to satisfy the positivity condition that $H(t) > 0$ and $P(t) \geq 0$ for all $t \geq 0$. System (1) has the following dynamical properties.

System (1) admits a unique positive equilibrium $E^*(H^*, P^*)$ where (see [21])

$$H^* = \frac{(r_1 - c_1) a_2}{a_1 (r_2 - c_2) + a_2 b_1}; P^* = \frac{(r_1 - c_1) (r_2 - c_2)}{a_1 (r_2 - c_2) + a_2 b_1}.$$

The positive equilibrium $E^*(H^*, P^*)$ obviously satisfies the equalities

$$r_1 - c_1 = a_1 P^* + b_1 H^*; r_2 - c_2 = a_2 \frac{P^*}{H^*}.$$

In fact, it can be shown easily that system (1) also admits another equilibrium, i.e. the boundary equilibrium $E^0(H^0, 0)$ where $H^0 = (r_1 - c_1)/b_1$.

It is obvious that if the number of predator is initially zero, i.e., $P(0) = 0$, then $P(t) = 0$ for all $t > 0$ and $H(t)$ becomes a logistic equation where its exact solution
is given by

\[
H(t) = \frac{H(0) (r_1 - c_1) \exp(-(r_1 - c_1)t)}{(r_1 - c_1) + b_1 H(0) (\exp(-(r_1 - c_1)t) - 1)}.
\]

Clearly that if \(t \to \infty\) then \(H(t) \to (r_1 - c_1)/b_1\). This shows that the boundary equilibrium is globally stable.

For the case of \(P(0) \neq 0\), Zhang et al. [21] proved that the positive equilibrium \(E^*(H^*, P^*)\) is asymptotically stable. It can also be checked via linearization technique that the boundary equilibrium is unstable.

3. Unconditionally positive NSFD scheme

To derive the numerical scheme, we first discretize the time variable \(t \geq 0\) at points \(t_n = nh (n = 0, 1, 2, \ldots)\) where \(h > 0\) is the time step size. The numerical solutions \(H\) and \(P\) at points \(t_n\) are denoted by \(H_n\) and \(P_n\), respectively. By applying the Mickens NSFD methodology to system (1), we obtain the following discrete Leslie-Gower model with harvesting

\[
H_{n+1} - H_n \frac{h}{h} = r_1 H_n - (a_1 P_n + b_1 H_n + c_1) H_{n+1}, \quad (2a)
\]

\[
P_{n+1} - P_n \frac{h}{h} = r_2 P_n - \left( a_2 \frac{P_n}{H_n} + c_2 \right) P_{n+1}. \quad (2b)
\]

The numerical scheme (2) is implicit due to the nonlocal approximation. After some simple algebraic manipulations, the discrete model (2) is transformed into the following explicit form

\[
H_{n+1} = \frac{(1 + hr_1) H_n}{1 + h (a_1 P_n + b_1 H_n + c_1)}, \quad (3a)
\]

\[
P_{n+1} = \frac{(1 + hr_2) P_n}{1 + h \left( a_2 \frac{P_n}{H_n} + c_2 \right)}. \quad (3b)
\]

Since all parameters in discrete model (3) are positive and if we assume that the initial values are \(H_0 > 0\) and \(P_0 \geq 0\) then obviously we get the following lemma.

**Theorem 3.1.** The NSFD method (2) or equivalently (3) is unconditionally positive.

It is easy to verify that the equilibria of discrete system (2) or (3) are exactly the same as those of continuous system (1), irrespective of \(h\); as stated in the following theorem.
Theorem 3.2. The NSFD method (2) or equivalently (3) has two equilibria, namely the boundary equilibrium \( E_0(H_0, 0) \) and the positive equilibrium \( E^*(H^*, P^*) \) which are independent of \( h \).

4. Stability of the equilibria

It is seen directly from discrete model (3) that if \( P_0 = 0 \) then \( P_n = 0 \) for all \( n > 0 \) and equation (3a) becomes

\[
H_{n+1} = \frac{(1 + hr_1)H_n}{1 + h(b_1H_n + c_1)}. \tag{4}
\]

Using the transformation \( H_n = \frac{1}{b_1} \left( \frac{y_{n+1}}{y_n} - (1 + hc_1) \right) \), equation (4) simplifies to a second order linear difference equation which can be solved easily, see [7] for the detail. The solution of (4) is

\[
H_n = \frac{H_0 (r_1 - c_1)}{H_0 b_1 + (r_1 - c_1 - H_0 b_1) (1 + hc_1)^n (1 + hr_1)^{-n}}. \tag{5}
\]

It can be checked that if \( n \to \infty \) then \( H_n \to (r_1 - c_1) / b_1 \). This shows that the solution of discrete model (4) is convergent to the boundary equilibrium whenever \( P_0 = 0 \); meaning that the boundary equilibrium is globally stable.

To verify the stability of each equilibrium for the case of \( P_0 \neq 0 \), we first define the following functions

\[
f(H, P) = \frac{(1 + hr_1)H}{1 + h(a_1P + b_1H + c_1)}, \tag{6a}
\]

\[
g(H, P) = \frac{(1 + hr_2)P}{1 + h(a_2P + c_2)} \tag{6b}
\]

and then consider the linearization of discrete system (3) at each equilibrium. The Jacobian matrix of the linearized system around equilibrium \( \left( \hat{H}, \hat{P} \right) \) is obviously given by

\[
J \left( \hat{H}, \hat{P} \right) = \begin{pmatrix}
\frac{\partial f(\hat{H}, \hat{P})}{\partial H} & \frac{\partial f(\hat{H}, \hat{P})}{\partial P} \\
\frac{\partial g(\hat{H}, \hat{P})}{\partial H} & \frac{\partial g(\hat{H}, \hat{P})}{\partial P}
\end{pmatrix} \tag{7}
\]

where

\[
\frac{\partial f(\hat{H}, \hat{P})}{\partial H} = \frac{1 + hr_1}{1 + h(a_1P + b_1H + c_1)} - \frac{hb_1(1 + hr_2)}{(1 + h(a_1P + b_1H + c_1))^2} \hat{H},
\]

\[
\frac{\partial f(\hat{H}, \hat{P})}{\partial P} = \frac{hr_1}{a_1 + b_1H + c_1}.
\]
An equilibrium \( \hat{H}, \hat{P} \) is stable if all eigenvalues \( \lambda_i, i = 1, 2 \) of the Jacobian matrix (7) satisfy \( |\lambda_i| < 1 \).

At the boundary equilibrium \( E^0(H^0, 0) \), the Jacobian matrix (7) can be written as

\[
J(H^0, 0) = \begin{pmatrix}
\frac{1+hc_1}{1+hr_1} & \frac{-ha_1(r_1-c_1)}{1+hr_1} \\
0 & \frac{1+hr_2}{1+hc_2}
\end{pmatrix}.
\]

The eigenvalues of \( J(H^0, 0) \) are \( \lambda_1 = \frac{1+hc_1}{1+hr_1} \) and \( \lambda_2 = \frac{1+hr_2}{1+hc_2} \). Since \( 0 < c_i < r_i \), \( i = 1, 2 \); it is obvious that \( 0 < \lambda_1 < 1 \) and \( \lambda_2 > 1 \) for any \( h \). Hence we have the following theorem about the stability of boundary equilibrium.

**Theorem 4.1.** The boundary equilibrium \( E^0 \) of discrete model (2) or equivalently (3) is unstable for all \( h \).

When analyzing the stability of positive equilibrium of discrete model (2), we will make use of the following lemma [1, 7].

**Lemma 4.2.** Roots of the quadratic equation \( \lambda^2 - A\lambda + B = 0 \) satisfy \( |\lambda_i| < 1 \), \( i = 1, 2 \) if and only if the following three conditions hold

\begin{enumerate}
    \item (1) \( 1 + A + B > 0 \)
    \item (2) \( 1 - A + B > 0 \)
    \item (3) \( B < 1 \)
\end{enumerate}

**Theorem 4.3.** The positive equilibrium \( E^* \) of discrete model (2) or equivalently (3) is asymptotically stable for all \( h \).

**Proof.** Substituting the positive equilibrium \( E^* \) into the Jacobian matrix (7) leads to

\[
J(H^*, P^*) = \begin{pmatrix}
1 - \frac{ha_1(r_1-c_1)}{(1+hr_1)(a_1(r_2-c_2)+a_2b_1)} & \frac{-ha_1(r_1-c_1)}{h(r_2-c_2)^2} \\
ha_2b_1(r_2-c_1) & \frac{1+hc_2}{1+hr_2}
\end{pmatrix}.
\]

The characteristics equation of matrix \( J(H^*, P^*) \) is \( \lambda^2 - A\lambda + B = 0 \) where
Asymptotically stable. This behavior is clearly seen in Figure 1. It is obvious that $c_i < r_i$, $i = 1, 2$, it is clear that $A > 0$ and $B > 0$. Hence

$$1 + A + B > 0.$$ 

Furthermore, we can also show that

$$1 - A + B = \frac{h^2 (r_1 - c_1) (r_2 - c_2) (a_2 b_1 + r_2 - c_2) + ha_2 b_1 (r_2 - c_2)}{(1 + hr_1) (1 + hr_2) (a_1 (r_2 - c_2) + a_2 b_1)} > 0.$$ 

Finally, after some algebraic manipulations, $B$ can be simplified as

$$B = \frac{p_0 + p_1 h + p_2 h^2}{q_0 + q_1 h + q_2 h^2},$$

where

- $p_0 = q_0 = a_2 b_1 + a_1 (r_2 + c_2)$,
- $p_1 = a_1 (r_2 (r_1 - c_2) + c_2 (r_2 - c_2)) + a_2 b_1 (c_1 + c_2)$,
- $q_1 = a_1 (r_2 (r_1 - c_2) + r_2 (r_2 - c_2)) + a_2 b_1 (r_1 + r_2)$,
- $p_2 = a_1 r_1 r_2 (r_2 - c_2) + a_2 b_1 c_2 - a_1 c_1 (r_2 - c_2)^2$,
- $q_2 = a_1 r_1 r_2 (r_2 - c_2) + a_2 b_1 r_1 r_2$.

It is obvious that $p_1 < q_1$ and $p_2 < q_2$ and therefore we have that

$$B < 1.$$ 

Hence all conditions in Lemma 4.2 are always satisfied for all $h$. It can be concluded that the positive equilibrium $E^*$ is asymptotically stable for any $h$. \qed

5. Numerical simulations

In this section we present numerical simulations to illustrate the analytical results obtained in the previous section. The parameters for these simulations are taken from [21], i.e., $r_1 = 1.6$, $r_2 = 1.0$, $a_1 = 0.7$, $a_2 = 0.25$, $b_1 = 0.3$, $c_1 = 0.821$, $c_2 = 0.859$. Using those parameters we have boundary equilibrium $E^0 (2.5967, 0.0)$ and positive equilibrium $E^* (1.1207, 0.6325)$. According to the previous analysis, $E^0$ is globally stable whenever $P (0) = 0$. If $P (0) \neq 0$ then $E^0$ is unstable while $E^*$ is asymptotically stable. This behavior is clearly seen in Figure 1.
Figure 1. Solutions \((H_n, P_n)\) of discrete model (2) or equivalently (3). If \(P_0 = 0\) then the solutions converge to \(E^0\); otherwise, the solutions converge to \(E^*\).

6. Conclusion

In this paper, we have proposed a discrete-time analogue of the harvesting Leslie-Gower predator-prey model by using the NSFD methodology. It is shown that the discrete model preserves some important features of the associated original continuous model such as positivity of solutions, equilibria and their stability. If there is no predator in the beginning, then the boundary equilibrium is globally asymptotically stable. If the predator is not initially zero, then the boundary equilibrium is unstable and the positive equilibrium is locally asymptotically stable. Such properties are independent of time-step size \(h\). This shows that the constructed NSFD scheme is dynamically consistent with its corresponding continuous model. In the future, we should investigate the global stability of the positive equilibrium of the constructed discrete model.

References

[8] T. Fayeldi, A. Suryanto and A. Widodo, Dynamical behaviors of a discrete SIR epi-