Dynamics of Harvested Predator-Prey System with Disease in Predator and Prey in Refuge

P.I. Trisdiani, Trisilowati* and A. Suryanto

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Brawijaya University, Jl. Veteran - Malang 65145 (Indonesia)

*Email: trisilowati@ub.ac.id

ABSTRACT

In this paper, a mathematical model of predator prey incorporating harvesting and disease in the predator and prey in refuge are developed and discussed. The harvesting of susceptible and infective predator is assumed to obey the Holling functional responses of type II and type I, respectively. According to the analysis, there exist five equilibrium points. The local stability property of each equilibrium point is presented. The analytical finding is then confirmed by some numerical simulations. Furthermore, we also investigate the effects of predator harvesting and prey refuge in the proposed model. It is found numerically that harvesting of susceptible predator using Holling functional responses type II can maintain the existence of all populations, and harvesting of infected predator can be used as a biological control to prevent the spread of the disease. Moreover, the prey in refuge can avoid the extinction of prey population.

Keywords: Disease, Holling functional responses, predator harvesting, prey refuge, stability analysis.

Mathematics Subject Classification: 92B05

Computing Classification System: G.1.7, J.3

1. INTRODUCTION

Predator prey system has attracted interest from many scientific communities including mathematics and biology. The effect of infectious diseases caused by viruses and bacteria is an important issue as well as ecological point view. The first eco-epidemiology model was proposed by [Anderson and May, 1982]. In this model, the Kermack and McKendrick epidemiological model is combined with the classical Lotka-Volterra predator-prey model. Based on this model, many authors have studied the influence of a disease in prey on the dynamical properties of predator-prey system; see e.g. [Chattopadhyay and Arino, 1999; Hethcote et al., 2004; Xiao and Chen, 2001; Rahman and Chakravarty, 2013]. The effects of a disease in predator on predator-prey model have been considered by [Han et al., 2001; Das, 2011].

Besides a disease, harvesting can significantly affect the dynamics of predator-prey system. Harvesting can not only decrease the population of prey or predator [Clark, 1976] but can also be considered as stabilizing factor [Goh et al., 1974], a destabilising factor [Azar et al., 1995] or even induce oscillating behaviour [Costa, 2007]. Bairagi et al. (2009) have studied the combined effects of
harvesting and disease in prey in a predator-prey model. They showed that harvesting can control the
spread of disease in a prey subpopulation. On the other hand, Cheve et al. (2010) have included
harvesting and disease in predator in a predator-prey model and concluded that harvesting may
prevent the spread of infectious diseases, so the resilience and stability of ecological systems will be
secure.

Interactions between predator and prey may reduce the prey population. However, prey can avoid
predator with some strategies, e.g. refuge to prevent prey extinction. In this respect, Wuhaib and
Hasan (2013) have incorporated the prey refuge in the model of Cheve et al. (2010). They concluded
that the prey refuge ensure the continuity and sustainability of all populations and can also control the
disease. In Wuhaib and Hasan (2013), it is assumed that the harvesting of predator populations, both
susceptible and infective predator follow the Holling functional responses of type I which is associated
with a bilinear function. To be more realistic, Lenzini and Rebaza (2010) proposed Holling functional
response type II to harvest predator. This function can explain a rational harvesting rate when
predator is considered to have economic value. By considering that the susceptible predator has such
economic value, in this paper, we propose a predator-prey model with prey refuge and disease and
harvesting in predator where the harvesting of susceptible predator is of the form of Holling functional
response type II.

2. MATHEMATICAL MODEL

In this section we reconsider a predator-prey model proposed by Wuhaib and Hasan (2013). By
assuming that the harvesting of susceptible predator obeys the Holling functional response type II, we
obtain the following model:

\[
\begin{align*}
\frac{dS}{dt} &= \beta(1-m)NS - \mu S - \gamma SI - \frac{hS}{c + S} \\
\frac{dI}{dt} &= \beta(1-m)NI - dI + \gamma SI - eI,
\end{align*}
\]

where \(N, S\) and \(I\) denote the prey, susceptible predator and infective predator population,
respectively. The parameter \(r, K, \alpha, m, \beta, \mu, \gamma, h, c, d\) and \(e\) are all positive constants. \(r\) and \(K\)
represents the intrinsic growth rate and carrying capacity of the prey, respectively. \(\alpha\) is the predation
rate, \(\beta\) is the conversion rate of predator, \(m\) is the prey refuge constant with \(m \in [0,1]\). We assume
that the parasite attack the predator population only with \(\gamma\) is the infection rate. The constant \(\mu\) is
the mortality rate of predator and \(d\) denotes an extra mortality due to infection. Accordingly, we
assume that \(\mu < d\). \(h\) and \(e\) are the harvest rate of susceptible predator and infective predator,
respectively and \( c \) is the half saturation constant. The initial conditions of (1) are given by \( N(0) > 0, S(0) > 0, I(0) > 0 \).

3. EQUILIBRIUM POINTS AND STABILITY ANALYSIS

3.1. Equilibrium Points

The equilibrium points are obtained by solving \( \frac{dN}{dt} = \frac{dS}{dt} = \frac{dI}{dt} = 0 \). It is found that the system has the following equilibrium points:

i. The trivial equilibrium point \( E_0 = (0,0,0) \) and the predator free equilibrium point \( E_1 = (K,0,0) \) which exist for all parametric values.

ii. The endemic equilibrium point \( E_2 = (N^*_2,0,I^*_2) \) is the extinction of susceptible predator, where

\[
N^*_2 = \frac{d + e}{\beta(1 - m)} \quad \text{and} \quad I^*_2 = \frac{r \left( 1 - \frac{N^*_2}{K} \right)}{\alpha(1 - m)},
\]

\( E_2 \) exist only if \( K > N^*_2 \).

iii. The disease free equilibrium point \( E_3 = (N^*_3,S^*_3,0) \) which is the extinction of infected predator,

\[
S^*_3 = \frac{r \left( 1 - \frac{N^*_3}{K} \right)}{\alpha(1 - m)} \quad \text{and} \quad N^*_3 \text{ is the positive root of equation } a_1 N^*_3^2 - a_2 N^*_3 + a_3 = 0, \text{where}
\]

\( a_1 = r\beta(1 - m), a_2 = \alpha\beta c K (1 - m)^2 + r\beta K (1 - m) + \mu r, a_3 = \alpha\mu c K (1 - m) + h\alpha K (1 - m) + \mu r K. \)

\( E_3 \) exist if \( K > N^*_3 \) and \( a_2^2 - 4a_1a_3 > 0 \).

iv. The interior equilibrium point \( E_4 = (N^*_4,S^*_4,I^*_4) \) where all population coexists. Here, \( N^*_4 \) is the positive root of equation

\[
b_1 N^*_4^2 - b_2 N^*_4 + b_3 = 0, \quad \text{where}
\]

\( b_1 = \beta r(1 - m), \)

\( b_2 = r\gamma (K\beta(1 - m) + yc + d + e) - \alpha K\beta(1 - m)^2(d + e - \mu), \)

\( b_3 = r\gamma (yc + d + e) - \alpha K(1 - m)(d + e - \mu)(yc + d + e) + h\gamma(1 - m), \)

\( S^*_4 = \frac{-\beta(1 - m)N^*_4 + d + e}{\gamma}, \text{ and } I^*_4 = \frac{\beta(1 - m)N^*_4 - \mu - \frac{h}{c + S^*_4}}{\gamma}. \)

The existence conditions of interior equilibrium point are

\[
\frac{\mu + \frac{h}{c + S^*_4}}{\beta(1 - m)} < N^*_4 < \min \left\{ K, \frac{d + e}{\beta(1 - m)} \right\},
\]

49
3.2. Stability Analysis

In this section, we analyze the local stability of each equilibrium point by considering the linearized system around the corresponding equilibrium point. The Jacobian matrix at equilibrium point \( \left( N^*, S^*, I^* \right) \) is given by

\[
J = \begin{bmatrix}
    r - \frac{2nN^*}{K} - \alpha(1-m)(S^* + I^*) & -\alpha(1-m)N^* & -\alpha(1-m)N^* \\
    \beta(1-m)S^* & \beta(1-m)N^* - \mu - h^* & -\frac{hc}{(c+S)^2} - \gamma S^* \\
    \beta(1-m)I^* & h^* & \beta(1-m)N^* - d + \gamma S^* - e
\end{bmatrix}
\]

Equilibrium point \( \left( N^*, S^*, I^* \right) \) is stable if the real part of all eigenvalues of the Jacobian matrix is negative.

**Theorem 1.** The trivial equilibrium point \( E_0 \) is always unstable. The system (1) is locally asymptotically stable around

1. \( E_1 \) if \( \beta(1-m)K < \min\{\mu + \frac{h}{c}, d + e\} \),

2. \( E_2 \) if \( \beta(1-m)N_2^* < \mu + h^* + \frac{h}{c} \),

3. \( E_3 \) if \( \beta(1-m)N_3^* + \gamma S_3^* < (d + e), \frac{rh}{(c+S)^2} - \alpha\beta(1-m)^2 < 0 \), and \( \frac{rS_3^*}{K} - \frac{hS_3^*}{(c+S)^2} > 0 \),

4. \( E_4 \) if \( \frac{(c+S_4^*)^2}{K} > \max\left\{\frac{hS_4^*}{N_4^*} - \frac{\alpha\beta(1-m)^2}{c}, \frac{hS_3^*}{N_3^*}, \frac{(c+S)^2}{N_4^*} \right\} \) and \( H_2 > 0 \), where \( H_2 \) is given in the proof.

**Proof.** The Jacobian matrix at \( E_0 \) is

\[
J(E_0) = \begin{bmatrix}
    r & 0 & 0 \\
    0 & -\mu - \frac{h}{c} & 0 \\
    0 & 0 & -d - e
\end{bmatrix},
\]

where the eigenvalues are \( \lambda_1 = r > 0, \lambda_2 = -\mu - \frac{h}{c} < 0, \) and \( \lambda_3 = -(d + e) < 0 \), and so \( E_0 \) is always unstable.
The Jacobian matrix at $E_1$ is

$$J(E_1) = \begin{bmatrix} -r & -\alpha(1-m)K & -\alpha(1-m)K \\ 0 & \beta(1-m)K - \frac{h}{c} & 0 \\ 0 & 0 & \beta(1-m)K - d - e \end{bmatrix}.$$ 

The eigenvalues are $\lambda_1 = -r < 0$, $\lambda_2 = \beta(1-m)K - \frac{h}{c}$ and $\lambda_3 = \beta(1-m)K - (d + e)$. It is clear that $E_1$ is asymptotically stable if $\beta(1-m)K < \min \{ \mu + \frac{h}{c}, d + e \}$.

(2). The Jacobian matrix at $E_2$ is

$$J(E_2) = \begin{bmatrix} -\frac{rN_2^*}{K} & -\alpha(1-m)N_2^* & -\alpha(1-m)N_2^* \\ 0 & \beta(1-m)N_2^* - \mu - \gamma_2^* - \frac{h}{c} & 0 \\ \beta(1-m)N_2^* & \gamma_2^* & 0 \end{bmatrix}.$$ 

The eigenvalues of $J(E_2)$ are $\lambda_1 = \beta(1-m)N_2^* - \mu - \gamma_2^* - \frac{h}{c}$ and

$$\lambda_{2,3} = \frac{1}{2} \left( -\frac{rN_2^*}{K} + \frac{\left(\frac{rN_2^*}{K}\right)^2}{4\alpha\beta(1-m)^2 N_2^* I_2^*} \right).$$

Then $E_2$ is asymptotically stable if $\beta(1-m)N_2^* < \mu + \gamma_2^* + \frac{h}{c}$.

(3). The Jacobian matrix at $E_3$ is

$$J(E_3) = \begin{bmatrix} -\frac{rN_3^*}{K} & -\alpha(1-m)N_3^* & -\alpha(1-m)N_3^* \\ \beta(1-m)S_3^* & hN_3^* & -\gamma S_3^* \\ 0 & 0 & \beta(1-m)N_3^* + \gamma S_3^* - (d + e) \end{bmatrix}.$$ 

The Jacobian matrix $J(E_3)$ has eigenvalues $\lambda_1 = \beta(1-m)N_3^* + \gamma S_3^* - (d + e)$, and

$$\lambda_{2,3} = \frac{1}{2} \left( -\frac{rN_3^*}{K} - \frac{hS_3^*}{c + S_3^*} \right) \pm \sqrt{\left(\frac{rN_3^*}{K} - \frac{hS_3^*}{c + S_3^*}\right)^2 + 4N_3^* S_3^* \left(\frac{r}{c + S_3^*} - \alpha\beta(1-m)^2\right)},$$

therefore, $E_3$ is asymptotically stable if the following conditions are satisfied.
(a) \( \beta(1-m)N_3^* + \gamma S_3^* < (d + e) \),

(b) \( \left( \frac{rh}{c + S_3^*} \right) - \alpha \beta(1-m)^2 < 0 \),

(c) \( \left( \frac{rN_3^*}{K} - \frac{hS_3^*}{(c + S_3^*)^2} \right) > 0 \).

(4). The Jacobian matrix at \( E_4 \) is

\[
J(E_4) = \begin{bmatrix}
v_{11} & v_{12} & v_{13} \\
v_{21} & v_{22} & v_{23} \\
v_{31} & v_{32} & 0
\end{bmatrix},
\]

where

\[
v_{11} = -\frac{rN_4^*}{K}, \quad v_{12} = -\alpha(1-m)N_4^*, \quad v_{13} = -\alpha(1-m)N_4^*, \quad v_{21} = \beta(1-m)S_4^*, \quad v_{22} = \frac{hS_4^*}{(c + S_4^*)}, \quad v_{23} = -\gamma S_4^*, \quad v_{31} = \beta(1-m)I_4^*, \quad v_{32} = -\gamma I_4^*.
\]

The characteristic equation of the Jacobian matrix in this case is given by

\[
\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0
\]

where

\[
A_1 = \frac{rN_4^*}{K} - \frac{hS_4^*}{(c + S_4^*)}, \quad A_2 = \gamma^2 S_4^* I_4^* + \alpha \beta(1-m)^2 N_4^* \left( S_4^* + I_4^* \right) - \frac{rhN_4^* S_4^*}{K \left( c + S_4^* \right)^2}, \quad \text{and}
\]

\[
A_3 = N_4^* S_4^* I_4^* \left( \frac{r^2}{K} - \frac{\alpha \beta(1-m)^2}{(c + S_4^*)^2} \right).
\]

Based on the Routh Hurwitz criterion, it is concluded that all eigenvalues of the characteristic equation (2) have negative real part if \( A_1 > 0, A_2 > 0 \) and \( A_1A_2 - A_3 > 0 \). Then \( E_4 \) is locally asymptotically stable if

(a) \( \frac{r(c + S_4^*)^2}{K} > \max \left\{ \frac{hS_4^*}{N_4^*}, \frac{\alpha \beta(1-m)^2}{\gamma^2} \right\} \),

(b) \( H_2 > 0 \), where

\[
H_2 = -\frac{rhN_4^* S_4^*}{K \left( c + S_4^* \right)^2} \left( \frac{rN_4^*}{K} - \frac{hS_4^*}{(c + S_4^*)^2} \right) + \frac{r \alpha \beta(1-m)^2 N_4^* \left( S_4^* + I_4^* \right)}{K} - \frac{hS_4^*}{(c + S_4^*)^2} \left( \gamma^2 I_4^* + \alpha \beta(1-m)^2 N_4^* \right).
\]

52
4. NUMERICAL SIMULATIONS

To illustrate our previous analytical results, we present some numerical simulations using the fourth-order Runge-Kutta method. The parameters for those simulations are

1) $r = 1.5$, $K = 10$, $\alpha = 0.7$, $\beta = 0.6$, $m = 0.95$, $\mu = 0.35$, $\gamma = 0.05$, $h = 0.2$, $c = 0.1$, $d = 0.45$, and $e = 0.1$.

2) $r = 1.5$, $K = 10$, $\alpha = 0.7$, $\beta = 0.6$, $m = 0.85$, $\mu = 0.25$, $\gamma = 0.1$, $h = 0.2$, $c = 0.1$, $d = 0.35$, and $e = 0.25$.

3) $r = 1.5$, $K = 10$, $\alpha = 0.7$, $\beta = 0.6$, $m = 0.5$, $\mu = 0.1$, $\gamma = 0.1$, $h = 0.1$, $c = 0.05$, $d = 0.25$, and $e = 0.3$.

4) $r = 0.5$, $K = 1$, $\alpha = 0.4$, $\beta = 0.3$, $m = 0.15$, $\mu = 0.0025$, $\gamma = 0.08$, $h = 0.001$, $c = 0.0005$, $d = 0.05$, and $e = 0.05$.

5) $r = 0.5$, $K = 1$, $\alpha = 0.4$, $\beta = 0.3$, $m = 0.5$, $\mu = 0.0025$, $\gamma = 0.08$, $h = 0.05$, $c = 5$, $d = 0.05$, and $e = 0.05$.

**Case 1:** It can be shown that the system has two equilibrium points, i.e. $E_0 = (0,0,0)$ and $E_1 = (10,0,0)$. The predator free equilibrium point $E_1$ is asymptotically stable which can be observed from Figure 1a, where all trajectories are convergent to $E_1$. This simulation shows that prey can grow without any interruption and reach its carrying capacity.

**Case 2:** The system has five equilibrium points, i.e. $E_0 = (0,0,0)$, $E_1 = (10,0,0)$, $E_2 = (6.1111,0.5556)$, $E_{31} = (9.85,0.2142,0)$ and $E_{32} = (2.9977,10.0032,0)$. In our previous analysis says that $E_2$ is the only equilibrium which is asymptotically stable. Such stability properties is clearly seen in Figure 1b. It shows that at the final stage, all predator become infective.

**Case 3:** In this case, the system has also five equilibrium points, i.e. $E_0 = (0,0,0)$, $E_1 = (10,0,0)$, $E_2 = (1.8333,0.35)$, $E_{31} = (0.4135,4.1085,0)$ and $E_{41} = (1.4514,1.1457,2.5180)$ . The only equilibrium point which is asymptotically stable is the disease free equilibrium point $E_{31}$, see Figure 1c. This figure shows that finally the disease disappears from the predator population.

**Case 4:** The system has seven equilibrium points, i.e. $E_0 = (0,0,0)$, $E_1 = (10,0,0)$, $E_2 = (0.3922,0.8939)$, $E_{31} = (0.9976,0.0035,0)$, $E_{32} = (0.0125,1.4522,0)$, $E_{41} = (0.3795,0.0403,0.8722)$ and $E_{42} = (0.1841,0.6633,0.5366)$ . The interior equilibrium point $E_{42}$ is the only asymptotically stable equilibrium. For illustration, see Figure 1d.
To show the effects of harvesting in infected predator, we perform simulations using parameters as in case 4 but with $e = 0$ as indicated in Figure 2a, then the solution convergent to $E_2 = (0.1961, 0.1182)$. Here, the disease becomes endemic and the number of susceptible predator goes to extinction. However, in case 4 where $e = 0.05$ ($e \neq 0$), it shows that harvesting of infected predator can prevent the spread of the disease. Furthermore, we show the influence of prey refuge. If a relatively large coefficient refuge ($m = 0.85$) is used then the solution is convergent to $E_{42} = (0.8557, 0.7687, 0.4338)$, which is coexist, see Figure 2b. However, if we consider that there is no prey refuge, i.e. $m = 0$, then the number of prey at the final stage is less than in the case if there exists prey refuge. Indeed, in the later case, the solutions of system is convergent to $E_{42} = (0.0391, 1.1229, 0.0848)$, see Figure 2c. Prey

Figure 1. Numerical simulation of model (1): (a) case 1; (b) case 2; (c) case 3; (d) case 4.
refuge can also maintain the existence of all populations.

![Figure 2](image_url)

**Figure 2.** Numerical solution of model (1) with parameter as in Case 4 but with (a) $e = 0$; (b) $m = 0.85$; (c) $m = 0$.

We perform a simulation using parameter in case 5 to show the effect of Holling functional responses type I and II on harvesting in susceptible predator. As seen in Figure 3a, the solution convergent to $(0.6667, 0, 0.8333)$. Here, if we use Holling functional response type I the population of susceptible predator extinct. However if we use Holling functional response type II as indicated in Figure 3b, the solution convergent to $(0.5606, 0.1989, 0.8996)$. It means that all population coexist. As we know that susceptible predator is considered to have economic value. Therefore, the use of Holling functional responses type II is more realistic than type I, because it can keep the existence of all populations, especially susceptible predator.
5. CONCLUSION

In this paper, we have investigated a mathematical model of predator-prey system including the interaction between harvesting and a disease in predator and prey in refuge. To describe this interaction, the model is comprised of three ordinary differential equations. The existence of equilibrium points as well as their stability properties are established. It is shown that the trivial equilibrium point is unstable, while other equilibrium points are stable under suitable conditions. Our results show that harvesting of susceptible predator using Holling functional responses type II can maintain the existence of all populations, and harvesting of infected predator can be used as a biological control to prevent the spread of the disease. Moreover, the prey in refuge can avoid the extinction of prey population.

6. REFERENCES


