Dynamics of a Predator-Prey Model Incorporating Prey Refuge, Predator Infection and Harvesting

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Abstract

We investigate the dynamics of a predator-prey model with infectious disease in the predator species. The population is divided into three classes, namely prey, susceptible predator and infective predator. We also consider a prey refuge and predator harvesting in the model. The predation rate and the growth rate due to predation of the infective predator are assumed to be smaller than those of susceptible predator. It is found that the model has five equilibrium points, i.e., the extinction of all population equilibrium, the predator extinction equilibrium, the susceptible predator extinction equilibrium, the infective predator extinction equilibrium and the coexistent equilibrium. The extinction of all population equilibrium is unstable, while other equilibrium points are asymptotically stable with certain condition. Such analytical finding is confirmed by some numerical simulations.

Keywords: Predator-Prey Model with Disease, Predator Harvesting, Prey Refuge

1 Introduction

One of the most important, interesting and challenging models in biological system are the interaction among species in the predator-prey relationship. There

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are many factors that influence the dynamics of predator-prey interactions. One of these factors is the hiding behavior of prey such that they are safe from their predator. Such refuges afford the prey some degree of protection from predation and simultaneously reduce the chance of extinction due to predation, see e.g., [7-9, 13]. Besides this effect, prey refuge may also lead to stabilizing (and destabilizing effect) on predator-prey models. Here, stabilization (destabilization) of stability refers to cases where an equilibrium point changes from an attractor (a repeller) to a repeller (an attractor) due to increase in the value of a control parameter [5, 7].

Besides refuge, infectious disease are also considered to have significant role in regulating population sizes of predator-prey system, see e.g., [1, 2, 6, 15]. On the other hand, by considering human needs, the exploitation of biological resources and the harvest of population are commonly practiced in fishery, forestry and wildlife management [4, 10]. Population harvesting generally has a strong effect on the dynamics of ecological system, including predator-prey system with disease. For example, Bairagi et.al. [2] examined the effects of disease and harvesting on the prey with Holling-type II response function. They showed that harvesting the prey can control the spread of the disease in the prey. Chevè et. al. [3] discussed the predator prey models with harvesting where only predator population is infective with disease. They found that if there is a susceptible predator, infective predator and prey in an ecosystem, then harvesting can be used to prevent infectious disease, so that the ecological resilience and stability will be maintained. Recently, Wuhaib and Abu Hasan [14] proposed a model to study the dynamics of infective predator with harvesting and refuge on the prey, i.e.

\[
\begin{align*}
\frac{dN}{dt} &= rN(1 - N) - aN(1 - m)(S + I), \\
\frac{dS}{dt} &= b(1 - m)NS - q_1S - vSI, \\
\frac{dI}{dt} &= b(1 - m)NI - q_2I + vSI,
\end{align*}
\]

where \( N, S, I \) are prey, susceptible predator, infective predator and \( v \) is the contact rate between the susceptible and infective predator. Here \( a \) is the predation rate and \( b \) is the predator growth rate \( (a > b) \). \( q_1 \) and \( q_2 \) are the harvest rates of the infective and susceptible predator respectively, while \( m \ (0 \leq m \leq 1) \) is a constant (which describes the ability of the prey to use constant refuge). They showed that constant refuge may lead to a situation which ensures the continuity of all populations and sustainability of the harvested species and controlling the disease such that endemic cannot occur. Later, Trisdiani et. al. [12] modified the model in [14] by assuming that the harvesting of susceptible predator is of the form of Holling functional response type II. They showed that the harvesting of susceptible predator can maintain the existence of all populations, and harvesting of infective predator can be used as a biological control to prevent the spread of the disease.
When considering a predator-prey system with infectious disease in the predator, it is naturally assumed that the infective predator has lower predation rate than that of susceptible predator. Consequently, the growth rate due to predation of susceptible predator will generally be higher than that of infective predator. To study the effects of different predation rates, the authors have modified model (1) by introducing different predation and different growth rates due to predation of susceptible and infective predator as follows [11]:

\[
\begin{align*}
\frac{dN}{dt} &= rN(1-N) - N(1-m)(a_1S + a_2I), \\
\frac{dS}{dt} &= b_1(1-m)NS - q_1S - vSI, \\
\frac{dI}{dt} &= b_2(1-m)NI - q_2I + vSI,
\end{align*}
\]

where \(a_1\) and \(a_2\) are the predation rate of susceptible and infective predator, respectively; \(b_1\) and \(b_2\) are respectively the growth rate (due to predation) of susceptible and infective predator. It is assumed that \(a_1 > a_2\) and \(b_1 > b_2\) with the initial condition \(N(0) > 0, S(0) > 0\) and \(I(0) > 0\). The authors have shown numerically that that different predation rate and predator growth rate may change the existence and stability of equilibrium point [10]. In this paper, we study the dynamical properties of model (2) analytically.

### 2 Equilibrium points: existence and stability

In this section we study the existence and stability behavior of model (2) at various equilibrium points. The equilibrium points of model (2) are:

(i). Trivial (The extinction of all population) equilibrium point: \(E_1(0,0,0)\).

(ii). The extinction of predator equilibrium point: \(E_2(1,0,0)\).

(iii). The extinction of susceptible predator equilibrium point: \(E_3(N_3,0,I_3)\)

where \(N_3 = \frac{q_2}{b_2(1-m)}\) and \(I_3 = \frac{r(1-N_3)}{a_2(1-m)}\).

(iv). The extinction of infective predator equilibrium point: \(E_4(N_4,S_4,0)\) where

\[
N_4 = \frac{q_1}{b_1(1-m)} \quad \text{and} \quad S_4 = \frac{r(1-N_4)}{a_1(1-m)}.
\]

(v). The coexistent equilibrium point: \(E_5(N_5,S_5,I_5)\) where

\[
\begin{align*}
N_5 &= \frac{(1-m)(a_1q_2 - a_2q_1) - vr}{(a_1b_2 - a_2b_1)(1-m)^2 - vr}, \\
S_5 &= \frac{q_2 - b_2(1-m)N_5}{v}, \quad \text{and} \\
I_5 &= \frac{b_1(1-m)N_5 - q_1}{v}.
\end{align*}
\]
We note that equilibrium points $E_1$ and $E_2$ always exist; $E_3$ exists if $N_3 < 1$ and $E_4$ exists if $N_4 < 1$; while the coexistent equilibrium $E_5(N_5, S_5, I_5)$ exists if one of the following conditions is satisfied:

(i). $m < \min \left\{ \frac{v_r}{a_1 q_2 - a_2 q_1}, 1 - \sqrt[2]{\frac{v_r}{a_1 b_2 - a_2 b_1}} \right\}$ and $m > \max \left\{ 1 - \frac{q_2}{b_2 N_5}, 1 - \frac{q_1}{b_1 N_5} \right\}$,

(ii). $m > \max \left\{ 1 - \frac{v_r}{a_1 q_2 - a_2 q_1}, 1 - \sqrt[2]{\frac{v_r}{a_1 b_2 - a_2 b_1}} \right\}$ and $m > \max \left\{ 1 - \frac{q_2}{b_2 N_5}, 1 - \frac{q_1}{b_1 N_5} \right\}$,

(iii). $1 - \frac{v_r}{a_1 q_2 - a_2 q_1} < m < 1 - \sqrt[2]{\frac{v_r}{a_1 b_2 - a_2 b_1}}$ and $m > \max \left\{ 1 - \frac{q_2}{b_2 N_5}, 1 - \frac{q_1}{b_1 N_5} \right\}$,

(iv). $1 - \sqrt[2]{\frac{v_r}{a_1 b_2 - a_2 b_1}} < m < 1 - \frac{v_r}{a_1 q_2 - a_2 q_1}$ and $m > \max \left\{ 1 - \frac{q_2}{b_2 N_5}, 1 - \frac{q_1}{b_1 N_5} \right\}$

The Jacobian matrix of model (2) at equilibrium $E^*(N^*, S^*, I^*)$ is given by:

$$J(E^*) = \begin{bmatrix}
a_{11} & -a_1(1-m)N^* & -a_2(1-m)N^* \\
b_1(1-m)S^* & a_{22} & -vS^* \\
b_2(1-m)I^* & vI^* & a_{33}
\end{bmatrix}$$

where $a_{11} = r(1-2N^*) - (1-m)(a_1 s^* + a_2 l^*)$, $a_{22} = b_1(1-m)N^* - q_1 - vl^*$ and $a_{33} = b_2(1-m)N^* + vS^* - q_2$. An equilibrium $E^*$ is (locally) asymptotically stable if the real parts of all eigenvalues of $J(E^*)$ are negative. The stability properties of all equilibrium points are summarized in the following theorem.

**Theorem 1.**

(i). The trivial equilibrium point $E_1$ is unstable.

(ii). The extinction of predator equilibrium point $E_2$ is asymptotically stable if $1 - m < \min \{q_1 / b_1, q_2 / b_2\}$.

(iii). The extinction of susceptible predator equilibrium point $E_3$ is asymptotically stable if $\frac{v_r(1-N_3)}{a_2(1-m)} > \frac{(b_1 q_2 - b_2 q_1)}{b_2}$.

(iv). The extinction of infective predator equilibrium point $E_4$ is asymptotically stable if $\frac{v_r(1-N_4)}{a_1(1-m)} < \frac{b_1 q_2 - b_2 q_1}{b_1}$.

(v). The coexistent equilibrium point $E_5$ is asymptotically stable if the following three conditions are satisfied:
\[ m < 1 - \frac{r(1 - 2N_s)}{(a_1S_5 + a_2I_5)}, \]  

(4)

\[ v(1 - m)^2N_sS_5I_5(a_2b_1 - a_1b_2) - v^2S_5I_5(r(1 - 2N_s) - (1 - m)(a_1S_5 + a_2I_5)) > 0, \] 

(5)

\[ [(1 - m)(a_1S_5 + a_2I_5) - r(1 - 2N_s)](1 - m)^2N_s(a_2b_2I_5 + a_1b_3S_5 + v^2S_5I_5) - [v(1 - m)^2N_sS_5I_5(a_2b_1 - a_1b_2) - v^2S_5I_5(r(1 - 2N_s) - (1 - m)(a_1S_5 + a_2I_5))] > 0 \)

(6)

**Proof:**

(i). The Jacobian matrix of model (2) at the trivial equilibrium point \( E_1(0,0,0) \) is

\[
J(E_1) = \begin{bmatrix}
  r & 0 & 0 \\
  0 & -q_1 & 0 \\
  0 & 0 & -q_2
\end{bmatrix}.
\]

The eigenvalues of \( J(E_1) \) are \( \lambda_1 = r > 0 \), \( \lambda_2 = -q_1 < 0 \), and \( \lambda_3 = -q_2 < 0 \). Consequently, the equilibrium point \( E_1 \) is unstable.

(ii). At the predator extinction equilibrium point \( E_2 \), the Jacobian matrix is given by

\[
J(E_2) = \begin{bmatrix}
  -r & -a_1(1-m) & -a_2(1-m) \\
  b_1(1-m) & -q_1 & 0 \\
  0 & 0 & b_3(1-m) - q_2
\end{bmatrix},
\]

which its eigenvalues are \( \lambda_1 = -r < 0 \), \( \lambda_2 = b_1(1-m) - q_1 \) and \( \lambda_3 = b_2(1-m) - q_2 \). Clearly that equilibrium \( E_2 \) will be stable if \( 1 - m < q_1/b_1 \) and \( 1 - m < q_2/b_2 \). In other words, the predator extinction equilibrium \( E_2 \) is stable whenever \( 1 - m < \min\{q_1/b_1, q_2/b_2\} \).

(iii). The Jacobian matrix at the susceptible predator extinction \( E_3 \) is given by

\[
J(E_3) = \begin{bmatrix}
  -\frac{rq_2}{b_2(1-m)} & -\frac{aq_2}{b_2} & -\frac{a_2q_2}{b_2} \\
  r & b_2 - q_2 & \frac{vr(1-N_3)}{a_2(1-m)} \\
  \frac{aq_2}{a_2} & \frac{vr(1-N_3)}{a_2(1-m)} & 0
\end{bmatrix},
\]

where \( \xi = (b_1q_2 - b_2q_1)/b_2 - vr(1-N_3)/(a_2(1-m)) \). The eigenvalues of \( J(E_3) \) are \( \lambda_1 = \xi \) and \( \lambda_{2,3} = \frac{1}{2}(-rN_3 \pm \sqrt{(rN_3)^2 - 4q_2r(1-N_3)}) \). It is clear that
\( \lambda_{2,3} < 0 \) and \( \lambda_1 < 0 \) only if \( \frac{vr(1 - N_3)}{a_2(1 - m)} > \frac{(b_q - b_2q_1)}{b_2} \). Thus, \( E_3 \) is stable if
\[
\frac{vr(1 - N_3)}{a_2(1 - m)} > \frac{(b_q - b_2q_1)}{b_2}.
\]

(iv). If \( E_4 \) is substituted into the Jacobian matrix then we have
\[
J(E_4) = \begin{bmatrix}
- rN_4 & - \frac{a_1q_2}{b_1} & - \frac{a_2q_1}{b_1} \\
\frac{b_1r(1 - N_4)}{a_1} & 0 & - \frac{vr(1 - N_4)}{a_1(1 - m)} \\
0 & 0 & \zeta
\end{bmatrix}.
\]

The eigenvalues of \( J(E_4) \) are
\[
\lambda_{1,2} = \frac{1}{2}(-rN_4 \pm \sqrt{(rN_4)^2 - 4a_1r(1 - N_4)}) < 0 \quad \text{and} \quad \lambda_3 = \zeta
\]
where
\[
\zeta = \frac{b_2q_1}{b_1} + \frac{vr(1 - N_4)}{a_1(1 - m)} - q_2.
\]
Hence, \( E_4 \) is asymptotically stable if \( \zeta < 0 \) or equivalently if
\[
\frac{vr(1 - N_4)}{a_1(1 - m)} < \frac{b_2q_1}{b_1} - \frac{b_q}{b_1}.
\]

(v). The Jacobian matrix at the coexistent equilibrium \( E_5 \) is
\[
J(E_5) = \begin{bmatrix}
(1 - m)(a_1S_5 + a_2I_5) - a_1(1 - m)N_5 & - a_2(1 - m)N_5 \\
\frac{b_1(1 - m)S_5}{a_1} & 0 & -vS_5 \\
\frac{b_2(1 - m)I_5}{a_1} & vI_5 & 0
\end{bmatrix}.
\]

The characteristic equation of \( J(E_5) \) is given by \( \lambda^3 + A\lambda^2 + B\lambda + C = 0 \), where
\[
A = (1 - m)(a_1S_5 + a_2I_5) - r(1 - 2N_3),
\]
\[
B = (1 - m)^2N_5(a_2b_2I_5 + a_1b_1S_5) + v^2S_5I_5,
\]
\[
C = v(1 - m)^2N_5S_5I_5(a_2b_1 - a_1b_2) - v^2S_5I_5(r(1 - 2N_5) - (1 - m)(a_1S_5 + a_2I_5)).
\]
According to the Routh-Hurwitz criterion, the coexistent equilibrium \( E_5 \) is asymptotically stable if \( A > 0, C > 0 \) and \( AB - C > 0 \) which are equivalent to conditions (4), (5) and (6). \( \square \)

3 Numerical simulations

To support our theoretical results, we perform some numerical simulations of system (2). For the first simulation, we take parameters \( r = 0.5, \ m = 0.5, \ a_1 = a_2 = 0.4, \ b_1 = b_2 = 0.43, \ q_1 = 0.125, \ q_2 = 0.2 \) and \( v = 0.08 \). In this case,
model (2) has five equilibrium points, i.e., \( E_1(0,0,0) \), \( E_2(1,0,0) \), \( E_3(0.930,0.0,0.174) \), \( E_4(0.581,1.046,0.0) \) and \( E_5(0.625,0.820,0.117) \). Notice that the predation rates of susceptible and infective predator are the same. The growth rate (due to predation) of susceptible predator is also the same as that of infective predator. It can be checked that only the coexistent equilibrium point \( E_5 \) satisfies the stability condition and other equilibrium points are unstable.

In Figure 1 we plot the solutions of model (2) using some different initial values. It can be seen that all solutions converge to \( E_5 \) as predicted by our theoretical results. Hence, the infection will be endemic in the predator population. Next, we consider model (2) with the same parameters as previous case except the predation rates and the growth rate of predators (due to predation), i.e., \( a_1 = 0.4 \), \( a_2 = 0.23 \), \( b_1 = 0.43 \) and \( b_2 = 0.12 \). Here we find that model (2) has only three common equilibrium points, i.e., \( E_1 \), \( E_2 \) and \( E_3 \).
equilibrium points, i.e., $E_1(0,0,0)$, $E_2(1,0,0)$ and $E_4(0.581,1.046,0.0)$, where $E_1$ and $E_2$ are unstable while $E_4$ is asymptotically stable. This behavior is clearly seen in Figure 2. It shows that reducing the value of predation rate and the growth rate (due to predation) of infective predator may lead to the extinction of infective predator. In other words, the disease disappear in the population of predator.

Finally, to see the effects of harvesting, we perform simulation using parameters as used in Figure 2 but with smaller value of harvest rate of infective predator, i.e., $q_2 = 0.1$. Reducing the value of harvest rate of infective predator will of course increase the total growth rate of infective predator ($dI/dt$) and the chance of infective predator extinction may be reduced. Indeed, in this case, model (2) has five equilibrium points where four of them are exactly the same as in the second simulation and the rest of them is the coexistent equilibrium $E_5(0.652,0.762,0.190)$. Based on our previous analytical results, it can be shown that $E_1, E_2, E_3$ and $E_4$ are unstable whereas $E_5$ is asymptotically stable. This phenomenon shows that the extinction of infective predator is prevented and prey, susceptible and infective predators co-exist in the system. The stability of $E_5$ can be seen from Figure 3, i.e., all solutions with different initial values are convergent to $E_5$.

![Figure 3](image-url)

**Figure 3.** Numerical solutions of model (2) with parameters $r = 0.5$, $m = 0.5$, $a_1 = 0.4$, $a_2 = 0.23$, $b_1 = 0.43$, $b_2 = 0.12$, $q_1 = 0.125$, $q_2 = 0.1$ and $v = 0.08$. All solutions with different initial values are convergent to $E_5$; showing that the disease is endemic and prey, susceptible and infective predators co-exist.

## 4 Conclusion

In this paper, a predator-prey model with prey refuge and predator infection and harvesting is discussed and analyzed. Particularly we derive the existence of all possible equilibrium points and their stability properties. There are five equili-
brium points, i.e., the extinction of prey and predator, the extinction of predator, the extinction of susceptible predator, the extinction of infective predator and the coexistent equilibrium point. It is found that the extinction of prey and predator is always unstable while others is conditionally asymptotically stable. It is also found that a smaller values of predation rate and growth rate (due to predation) of infective predator than those of susceptible predator may lead to the extinction of infective predator. However, reducing the harvesting rate of infective predator will increase the prevention of infective predator extinction such that all population survive.

References


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