Optimal Control of a Vector-Host Epidemic Model with Direct Transmission

D. E. Mahmudah, A. Suryanto* and Trisilowati

Department of Mathematics
Faculty of Sciences - Brawijaya University
Jl. Veteran Malang 65145, Indonesia

Copyright © 2013 D. E. Mahmudah, A. Suryanto and Trisilowati. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper we extend an existing vector-host epidemic model with direct transmission to assess the impact of two control measures, i.e. the preventive control to minimize vector-host contacts and the insecticide control to the vector. The aim is to derive optimal prevention strategies with minimal implementation cost. The characterization of optimal control is performed analytically by applying Pontryagin Minimum Principle. The obtained optimality system is then solved numerically to investigate that there are cost effective control efforts in reducing the incidence of infectious hosts and vectors.

Keywords: A vector-host epidemic model, Direct transmission, Optimal control, Pontryagin minimum principle, Forward-backward sweep method

1 Introduction

Many mathematical models have been used to understand the spread of vector-host epidemic diseases such as malaria, dengue, chikungunya, elephantiasis, and dysentery; see e.g. [5]. Various kinds of strategies for disease
control have been carried out with the aim of minimizing infective host or as prevention efforts. For example, optimal strategies for controlling the spread of malaria disease using prevention and treatment on malaria have been studied in [3]. Similarly, Agusto et al. [1] have investigated an optimal prevention strategies of malaria disease using treatment, insecticide treated bed nets and spray of mosquito insecticide as the system control variables. Okosun and Makinde [8] have studied the impact of drug resistance in malaria transmission to investigate the role of drug resistance individuals in malaria transmission and its optimal control problem.

Generally, insects or vectors become the main intermediaries of infectious disease to the host population; see e.g. [5] and [7]. In such vector-host epidemic model, transmission disease is usually caused by contacts between infective vectors and susceptible hosts only, see e.g. [1], [5], [7] and [9]. However, the direct transmission such as a contact between infective host and susceptible host population should be incorporated in the model [4]. By considering the direct transmission, in the normalized variables, Cai and Li [4] proposed the following vector-host epidemic model

\[
\begin{align*}
\frac{dS_H}{dt} &= \mu - \beta_1 S_H I_H - m \beta_2 S_H I_V + \phi I_H - \mu S_H \\
\frac{dI_H}{dt} &= m \beta_2 S_H I_V + \beta_1 S_H I_H - (\mu + \phi) I_H \\
\frac{dS_V}{dt} &= \eta - \beta S_V I_H - \eta S_V \\
\frac{dI_V}{dt} &= \beta S_V I_H - \eta I_V ,
\end{align*}
\]  

(1)

where \( S_H \) and \( I_H \) represents the proportion of susceptible and infective host population, respectively while \( S_V \) and \( I_V \) are respectively the proportion of susceptible and infective vector population. In model (1), \( \beta_1 \), \( \beta_2 \) and \( \beta \) are respectively the rate of direct transmission from infectious host to susceptible host, the rate of transmission from vector to host and the of transmission from host to vector; \( \mu \) is the host birth rate which is assumed to be equal to the host mortality rate; \( \eta \) is the vector birth rate which is assumed to be equal to the vector mortality rate; \( \phi \) is the recovery rate of infectious hosts; and \( m \) is the ratio of vector population and host population. Cai and Li [4] have shown that

\[
\Gamma = \{ (S_H, I_H, S_V, I_V) \in \mathbb{R}_+^4 : S_H, I_H, S_V, I_V \geq 0, S_H + I_H = 1, S_V + I_V = 1 \}
\]

is invariant region. It is also shown that system (1) has a disease free equilibrium \( E_0 = (1,0,1,0) \) which is asymptotically stable if \( R_0 < 1 \). Beside the disease free
equilibrium, system (1) has also an endemic equilibrium \( \hat{E} = (\hat{S}_H, \hat{I}_H, \hat{S}_V, \hat{I}_V) \) which is asymptotically stable whenever \( R_0 > 1 \), where

\[
\hat{S}_H = \frac{(\mu + \phi) \hat{I}_H}{\beta \beta_1 \hat{I}_H^2} + (\eta + \beta \hat{I}_H + \mu \eta), \quad \hat{S}_V = \frac{\eta}{\eta + \beta \hat{I}_H}, \quad \hat{I}_V = \frac{\beta \hat{I}_H}{\eta + \beta \hat{I}_H}.
\]

Here \( R_0 \) is the basic reproduction number which is defined by

\[
R_0 = \left( \frac{\beta \beta_2 m}{\mu + \phi} \right) \left( \frac{\beta}{\eta} \right) + \left( \frac{\beta_1}{\mu + \phi} \right),
\]

and \( \hat{I}_H \) is the root of the following quadratic equation

\[
a_2 (\hat{I}_H)^2 + a_1 \hat{I}_H + a_0 = 0,
\]

with \( a_2 = \beta \beta_1 > 0, a_1 = \eta \beta_1 + m \beta \beta_2 - \beta \beta_1 + (\mu + \phi) \beta, \) and \( a_0 = -\eta (\mu + \phi) (R_0 - 1) < 0 \).

In this paper, we extend model (1) by including the preventive contact control in the form of medical treatment to minimize vector host contacts as well as the level of larvacide and adulticide used for vectors control at vector’s habitat to minimize the infected host at minimal cost.

2 Optimal Control Problem

To extend the basic vector-host epidemic model with direct transmission (1), we introduce two control measures, i.e. the preventive contact control in the form of medical treatment to minimize vector host contacts \((u_1)\) and the level of larvacide and adulticide used for vector’s control at vector’s habitat \((u_2)\) as vector’s control with \( 0 \leq u_1 \leq 1 \) and \( 0 \leq u_2 \leq 1 \). By including these two control measures, we obtained the following model
Now, we consider an optimal control problem to minimize the objective functional

$$ J(u_1, u_2) = \int_0^T [BI_H(t) + Cu_1^2(t) + Du_2^2(t)] dt, $$

subject to system (2) where $T$ is the final time and all parameter $B$, $C$, and $D$ are positive balancing cost factors. Our aim is to minimize the number of infected host $I_H(t)$, while minimizing the cost for applying control $u_1(t)$ and $u_2(t)$. Thus, we seek an optimal control pair $u_1^*$ and $u_2^*$ such that

$$ J(u_1^*, u_2^*) = \min \{ J(u_1, u_2) | u_1, u_2 \in U \}, $$

where $U = \{(u_1, u_2) | u_i : [0, T] \rightarrow [0, 1], i = 1, 2\}$ is control set. For this purposes we shall use Pontryagin’s Maximum Principle. This principle converts (2), (3) and (4) into a problem of minimizing a Hamiltonian $(H)$, defined by

$$ H = BI_H(t) + Cu_1^2(t) + Du_2^2(t) $$

$$ + \gamma_{s_H} (\mu - \beta_1 S_H I_H - (1 - u_1)m\beta_2 S_H I_V + \phi I_H - \mu S_H) $$

$$ + \gamma_{I_H} (1 - u_1)m\beta_2 S_H I_V + \beta_1 S_H I_H - (\mu + \phi)I_H) $$

$$ + \gamma_{S_V} (\eta - \beta S_V I_H - \eta S_V - cu_2 S_V) + \gamma_{I_V} (\beta S_V I_H - \eta I_V - cu_2 I_V) $$

where $\gamma_{s_H} = \gamma_{s_H}(t), \gamma_{I_H} = \gamma_{I_H}(t), \gamma_{S_V} = \gamma_{S_V}(t)$, and $\gamma_{I_V} = \gamma_{I_V}(t)$ are costate variables. Applying Pontryagin’s Minimum Principle and the existence result of the optimal control, we obtain the following theorem.
Theorem 1. Given an optimal control \(u_1^*, u_2^*\) and solutions \(S_H^*, I_H^*, S_V^*, I_V^*\) of the corresponding state system (2) that minimizes \(J(u_1, u_2)\) over \(U\). Then there exists costate variables \(\gamma_{S_H}, \gamma_{I_H}, \gamma_{S_V}, \gamma_{I_V}\), such that

\[
\frac{d\gamma_{S_H}}{dt} = (\beta_1 I_H + (1-u_1)m\beta_2 I_V + \mu)\gamma_{S_H} + (- (1-u_1)m\beta_2 I_V - \beta_1 I_H)\gamma_{I_H}
\]

\[
\frac{d\gamma_{I_H}}{dt} = -B + (\beta_1 S_H - \phi)\gamma_{S_H} + (- \beta_1 S_H + \mu + \phi)\gamma_{I_H} + \beta S_V \gamma_{S_V} - \beta S_V \gamma_{I_V}
\]

\[
\frac{d\gamma_{S_V}}{dt} = (\beta I_H + \eta + cu_2)\gamma_{S_V} - \beta I_H \gamma_{I_V}
\]

\[
\frac{d\gamma_{I_V}}{dt} = (1-u_1)m\beta_2 S_H \gamma_{S_H} - (1-u_1)m\beta_2 S_H \gamma_{I_H} + (\eta + cu_2)\gamma_{I_V},
\]

with transversality conditions \(\gamma_{S_H}(T) = \gamma_{I_H}(T) = \gamma_{S_V}(T) = \gamma_{I_V}(T) = 0\), and the controls \(u_1^*\) and \(u_2^*\) satisfy the optimality condition,

\[
u_1^* = \min \left\{ \max \left( 0, \frac{\gamma_{I_H} - \gamma_{S_H}m\beta_2 S_H I_V}{2C} \right) \right\},
\]

\[
u_2^* = \min \left\{ \max \left( 0, \frac{\gamma_{S_V} cS_V + \gamma_{I_V} cI_V}{2D} \right) \right\}.
\]

Proof. The existence of optimal control can be proved using the same argument as in Theorem 1 of [2] which is based on the boundedness of solution of system (1) without control variables. The differential equations governing the costate variables are obtained by taking the derivative of the Hamiltonian function, evaluated at the optimal control. Then the costate system can be written as

\[
\frac{d\gamma_{S_H}}{dt} = -\frac{dH}{dS_H} = (\beta_1 I_H + (1-u_1)m\beta_2 I_V + \mu)\gamma_{S_H} + (- (1-u_1)m\beta_2 I_V - \beta_1 I_H)\gamma_{I_H}
\]

\[
\frac{d\gamma_{I_H}}{dt} = -\frac{dH}{dI_H} = -B + (\beta_1 S_H - \phi)\gamma_{S_H} + (- \beta_1 S_H + \mu + \phi)\gamma_{I_H} + \beta S_V \gamma_{S_V}
\]

\[
- \beta S_V \gamma_{I_V}
\]
\[
\begin{align*}
\frac{d\gamma_{SV}}{dt} &= -\frac{dH}{dS_V} = (\beta I_H + \eta + cu_2)\gamma_{SV} - \beta I_H \gamma_{IV}, \\
\frac{d\gamma_{IV}}{dt} &= -\frac{dH}{dI_V} = (1-u_1)m\beta_2 S_H \gamma_{SH} - (1-u_1)m\beta_2 S_H \gamma_{IH} + (\eta + cu_2)\gamma_{IV},
\end{align*}
\]

with transversality conditions \( \gamma_{sn}(T) = \gamma_{su}(T) = \gamma_{sv}(T) = \gamma_{iv}(T) = 0 \).

On the interior of the control set, where \( 0 \leq u_i \leq 1 \), for \( i = 1,2 \), we have

\[
0 = \frac{\partial H}{\partial u_i} = 2Cu_i + m\beta_2 S_H I_V \gamma_{SH} - m\beta_2 S_H I_V \gamma_{IH},
\]

\[
0 = \frac{\partial H}{\partial u_2} = 2Du_2 - \gamma_{sv} cS_v^* - \gamma_{iv} cI_v^*.
\]

Solving for the optimal control, we obtain

\[
u_1^* = \left( \frac{\gamma_{IH} - \gamma_{SH}}{2C} \right) m\beta_2 S_H I_V^*,
\]

\[
u_2^* = \frac{\gamma_{SV} cS_v^* + \gamma_{IV} cI_v^*}{2D}.
\]

Hence, the optimal controls are

\[
u_1^* = \min \left\{ \max \left( 0, \frac{\gamma_{IH} - \gamma_{SH}}{2C} m\beta_2 S_H I_V^* \right) , 1 \right\},
\]

\[
u_2^* = \min \left\{ \max \left( 0, \frac{\gamma_{SV} cS_v^* + \gamma_{IV} cI_v^*}{2D} \right) , 1 \right\}.
\]

3 Numerical Simulations

In this section, we investigate numerically the optimal solution to the vector-host epidemic model with direct transmission by Sweep Backward-Forward method [6]. The optimality system is solved using the fourth order Runge-
Optimal control of a vector-host epidemic model

Kutta. The state system is solved forward in time with initial conditions $S_H(0)=0.9, I_H(0)=0.1, S_V(0)=0.7$ and $I_V(0)=0.3$ while the costate system is solved backward in time. The controls values are update at the end of iteration using the formula for optimal controls. For the following numerical simulations, we use parameters given in Table 1.

Table 1. Values of parameters in the vector-host epidemic model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\mu$</th>
<th>$\phi$</th>
<th>$\eta$</th>
<th>$\beta$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.4427</td>
<td>0.9619</td>
<td>0.0046</td>
<td>0.7749</td>
<td>0.8173</td>
<td>0.8687</td>
<td>0.05</td>
</tr>
</tbody>
</table>

![Figure 1](image)

Figure 1. Numerical solution of system (1) with and without control.

Using parameter in Table 1, we find that the reproduction number is $R_0 = 11.6984 > 1$ which means that without control, any solution will be convergent to the endemic equilibrium. However, when control variables are introduced in system (1) then different behavior of solutions are obtained. For
example if we take the weight constant values in the objective functional are chosen to be $B = 1, C = 10, D = 10$, then we notice that the presence of controls may reduce the number of infective host as well as the number of infective vector significantly. The obtained optimal control $u_1^*(t)$ and $u_2^*(t)$ can be seen in Figure 2.

![Graphs showing optimal control profiles](image)

**Figure 2.** The optimal controls profile: (a) function $u_1^*(t)$ and (b) function $u_2^*(t)$.

## 4 Conclusion

In this work we have studied the effects of preventive control to reduce vector-host contacts and the insecticide control to the vector in a vector-host epidemic model with direct transmission. Using Pontryagin Minimum Principle we have established the existence of optimal control pair which minimize the proportion of both infective host and infective vector as well as minimize the cost of applying controls. The numerical simulations with and without control show that the control strategy helps to reduce the number of both infective host and infective vector significantly.

### Acknowledgment

Part of this research is financially supported by the Directorate General of Higher Education, Ministry of Education and Culture of Republic Indonesia via DIPA of Brawijaya University No.: DIPA-023.04.2.414989/2013, and based on letter of decision of Brawijaya University Rector No.: 295/SK/2013.
References


Received: July 25, 2013